

CONDUCTORS AND DIELECTRICS

الموصلات والعوازل

5.1 Current and Current Density

Electric charges in motion constitute a **current**. The unit of current is the ampere (A), defined as a rate of movement of charge passing a given reference point (or crossing a given reference plane) of one coulomb per second. Current is symbolized by I , and therefore

$$I = \frac{dQ}{dt}$$

We find the concept of **current density**, measured in amperes per square meter (A/m²), more useful. Current density is a vector represented by \mathbf{J} .

Total current is obtained by integrating,

$$I = \int_S \mathbf{J} \cdot d\mathbf{S}$$

Current density may be related to the **velocity** of **volume charge density** at a point.

$$\mathbf{J} = \rho_v \mathbf{U}$$

Where \mathbf{U} is velocity and ρ_v is volume charge density

This last result shows clearly that charge in motion constitutes a current. We call this type of current **convection current**

Example: Given the current density $\mathbf{J} = 10\rho^2 z \mathbf{a}_\rho - 4\rho \cos^2 \phi \mathbf{a}_\phi$ mA/m²: determine the total current flowing outward through the **circular** band $\rho = 3$, $0 < \phi < 2\pi$, $2 < z < 2.8$?

Solution:

$$I = \int_S \mathbf{J} \cdot d\mathbf{S} = \int \int (10\rho^2 z \mathbf{a}_\rho - 4\rho \cos^2 \phi \mathbf{a}_\phi) \cdot \rho d\phi dz \mathbf{a}_\rho$$

$$I = \int_2^{2.8} \int_0^{2\pi} 10\rho^3 z d\phi dz = 10(3)^2 \times 2\pi \times \left[\frac{z^2}{2} \right]_2^{2.8} = 3.26 \text{ mA}$$

Example: Find the current in the circular wire shown in figure below, if the current density $\mathbf{J}=15(1 - e^{-1000r}) \mathbf{a}_z$ A/m²: the radius of the wire is 2mm?

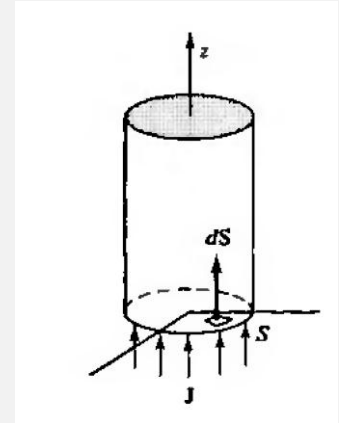
Solution:

$$I = \int_S \mathbf{J} \cdot d\mathbf{S}$$

$$d\mathbf{S} = \rho d\rho d\phi \mathbf{a}_z$$

$$I = \int_0^{2\pi} \int_0^{0.002} 15(1 - e^{-1000r}) \mathbf{a}_z \cdot \rho d\rho d\phi \mathbf{a}_z$$

$$I = \int_0^{2\pi} \int_0^{0.002} 15 \rho d\rho d\phi - \int_0^{2\pi} \int_0^{0.002} 15\rho e^{-1000r} \rho d\rho d\phi = 0.133 \text{ A}$$



Example: Find the total current outward directed from a 1m cube with one corner at the origin and edge parallel to the coordinate axes if $\mathbf{J}=2x^2\mathbf{a}_x + 2xy^3\mathbf{a}_y + 2xy\mathbf{a}_z$ A/m²?

Solution:

$$I = \int_S \mathbf{J} \cdot d\mathbf{S}$$

$$d\mathbf{S} = dx dy \mathbf{a}_z + dy dz \mathbf{a}_x + dz dx \mathbf{a}_y$$

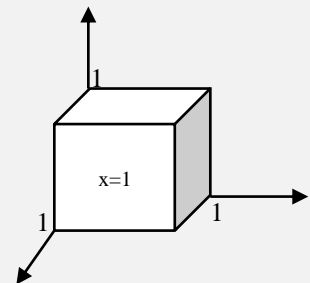
$$I = \iiint (2x^2\mathbf{a}_x + 2xy^3\mathbf{a}_y + 2xy\mathbf{a}_z) \cdot (dx dy \mathbf{a}_z + dy dz \mathbf{a}_x + dz dx \mathbf{a}_y)$$

$$I = \iiint 2x^2 dy dz + \iiint 2xy^3 dz dx + \iiint 2xy dx dy$$

$$(\text{at } x=1) - (\text{at } x=0) \quad (\text{at } y=1) - (\text{at } y=0) \quad (\text{at } z=1) - (\text{at } z=0)$$

$$I = \int_0^1 \int_0^1 2 dy dz - 0 + \int_0^1 \int_0^1 2x dz dx - 0 + \int_0^1 \int_0^1 2xy dx dy - \int_0^1 \int_0^1 2xy dx dy$$

$$I = 2[y]_0^1 [z]_0^1 + [x^2]_0^1 [z]_0^1 - 0 = 2 + 1 = 3 \text{ A}$$



5.2 Continuity of Current

The continuity equation follows when we consider any region bounded by a closed surface. The current through the closed surface is

$$I = \int_S \mathbf{J} \cdot d\mathbf{S}$$

and this *outward flow* of positive charge must be balanced by a decrease of positive charge (or perhaps an increase of negative charge) within the closed surface. If the charge inside the closed surface is denoted by Q_i , then the rate of decrease is $-dQ_i/dt$ and the principle of conservation of charge requires

$$I = \int_S \mathbf{J} \cdot d\mathbf{S} = -\frac{dQ_i}{dt}$$

$$\int_S \mathbf{J} \cdot d\mathbf{S} = \int_{vol} (\nabla \cdot \mathbf{J}) dv$$

We next represent the enclosed charge Q_i by the volume integral of the charge density

$$\int_{vol} (\nabla \cdot \mathbf{J}) dv = -\frac{d}{dt} \int_{vol} \rho_v dv$$

If we agree to keep the surface constant, the derivative becomes a partial derivative and may appear within the integral

$$\int_{vol} (\nabla \cdot \mathbf{J}) dv = \int_{vol} \frac{\partial \rho_v}{\partial t} dv$$

from which we have our point form of the continuity equation,

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$$

Remembering the physical interpretation of divergence, this equation indicates that the current, or charge per second, diverging from a small volume per unit volume is equal to the time rate of decrease of charge per unit volume at every point.

5.3 Metallic Conductors

Let us first consider the conductor. Here the valence electrons, or *conduction*, or *free*, electrons, move under the influence of an electric field. With a field \mathbf{E} , an electron having a charge $Q = -e$ will experience a force

$$\mathbf{F} = -e\mathbf{E}$$

In free space, the electron would accelerate and continuously increase its velocity (and energy); in the crystalline material, the progress of the electron is impeded by continual collisions with the thermally excited crystalline lattice structure, and a constant average velocity is soon attained. This velocity \mathbf{U}_d is termed the **drift velocity**, and it is linearly related to the electric field intensity by the **mobility** of the electron in the given material.

$$\mathbf{U}_d = -\mu_e \mathbf{E}$$

where μ_e is the mobility of an electron, mobility is measured in the units of square meters per volt-second; typical values are 0.0012 for aluminum, 0.0032 for copper, and 0.0056 for silver.

$$\mathbf{J} = -\rho_e \mu_e \mathbf{E}$$

where ρ_e is the free-electron charge density

The relationship between \mathbf{J} and \mathbf{E} for a metallic conductor, however, is also specified by the conductivity σ (sigma),

$$\mathbf{J} = \sigma \mathbf{E}$$

where σ is measured in Siemens per meter (S/m).

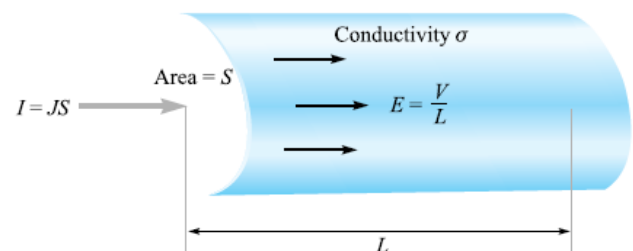
Conductivity may be expressed in terms of the charge density and the electron mobility,

$$\sigma = -\rho_e \mu_e$$

If a conductor of uniform cross-sectional area S and length L , as shown in Figure below, has a voltage difference V between its ends, then

Assume that \mathbf{J} and \mathbf{E} are *uniform*

$$I = \int_S \mathbf{J} \cdot d\mathbf{S} = JS \Rightarrow J = \frac{I}{S} = \sigma E$$



$$V_{AB} = - \int_B^A E \cdot dL = EL \Rightarrow E = \frac{V}{L}$$

$$\therefore \frac{I}{S} = \sigma \frac{V}{L}$$

$$\frac{V}{I} = \frac{L}{\sigma S}$$

The ratio of the **potential difference** between the **two ends of the cylinder** to the current entering the more positive end,

$$R = \frac{L}{\sigma S}$$

- When the fields are nonuniform,

$$R = \frac{V_{ab}}{I} = \frac{- \int_b^a E \cdot dL}{\int_S \sigma E \cdot dS}$$

Example: Find the resistance between the inner and outer curved surfaces of the block shown in

Fig. below, where the material is silver for which $\sigma = 6.17 \times 10^7$ S/m. if $\mathbf{J} = k/\rho \mathbf{a}_\rho$?

Solution:

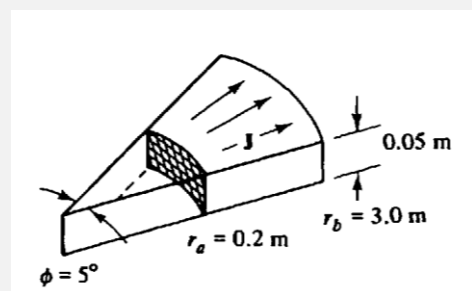
$$\mathbf{J} = \sigma \mathbf{E}$$

$$E = \frac{\mathbf{J}}{\sigma} = \frac{k}{\sigma \rho} \mathbf{a}_\rho$$

$$R = \frac{\int_{0.2}^3 \frac{k}{\sigma \rho} d\rho}{\int_0^{0.05} \int_0^{50} \frac{k}{\rho} \rho d\phi dz}$$

$$R = \frac{\ln \frac{3}{0.2}}{\sigma \times 0.0873 \times 0.005}$$

$$R = 1.01 \times 10^{-5} \Omega$$



5.4 Conductor Properties and Boundary Conditions

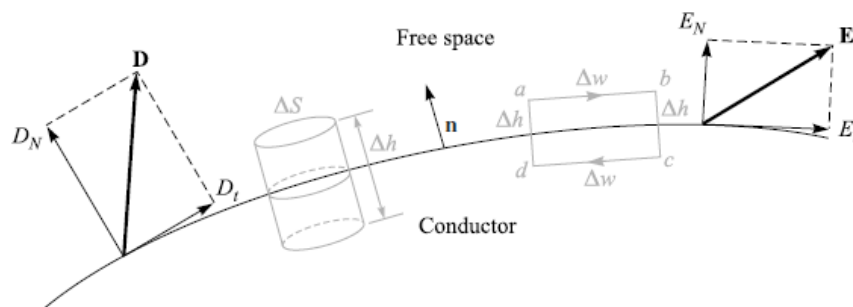
For electrostatics, no charge and no electric field may exist at any point within a conducting material. Charge may, however, appear on the surface as a surface charge density, and our next investigation concerns the fields external to the conductor.

To summarize the principles which apply to conductors in electrostatic fields, we may state that

1. *The static electric field intensity inside a conductor is zero.*
2. *The static electric field intensity at the surface of a conductor is everywhere directed normal to that surface.*
3. *The conductor surface is an equipotential surface.*

5.4.1 Conductor-Dielectric Boundary Conditions

Under static conditions all net charge will be on the outer surfaces of a conductor and both E and D are therefore zero within the conductor. Because the electric field is a conservative field, the line integral of $E \cdot dL$ is zero for any closed path. A rectangular path with corners 1, 2, 3, 4 is shown in Figure below.



$$\oint E \cdot dL = 0$$

around the small closed path $abcda$. The integral must be broken up into four parts

$$\int_a^b + \int_b^c + \int_c^d + \int_d^a = 0$$

If the path lengths b to c and d to a are now permitted to approach zero, keeping the interface between them, then the second and fourth integrals are zero.

The path from c to d is within the conductor where E must be zero. This leaves

$$\int_a^b E \cdot dL = \int_a^b E_t dl = 0$$

where E_t is the tangential component of E at the surface of the dielectric. Since the interval a to b can be chosen arbitrarily, at each point of the surface.

$$E_t = D_t = 0$$

To discover the conditions on the normal components, a small, closed, right circular cylinder is placed across the interface; Gauss' law applied to this surface gives

$$\oint D \cdot dS = Q_{enc}$$

$$\int_{top} D \cdot dS + \int_{bottom} D \cdot dS + \int_{side} D \cdot dS = \int_A \rho_S \cdot dS$$

The third integral is zero since, as just determined, $D_t = 0$ on either side of the interface. The second integral is also zero, since the bottom of the cylinder is within the conductor, where D and E are zero. Then,

$$\int_{top} D \cdot dS = \int_{top} D_n dS = \int_A \rho_S dS$$

$$D_n = \rho_S$$

$$E_n = \frac{\rho_S}{\epsilon}$$

The electric flux leaves the conductor in a direction normal to the surface, and the value of the electric flux density is numerically equal to the surface charge density.

Example: A solid conductor has a surface described by $x + y = 3\text{m}$ and extends toward the origin. At the surface the electric field intensity is 0.35 V/m . Express E and D at the surface and find ρ_s ?

Solution:

The unit vector normal to the surface is :

$$a_N = \frac{a_x + a_y}{\sqrt{2}}$$

$$E_N = 0.35 \frac{a_x + a_y}{\sqrt{2}} = 0.247(a_x + a_y)$$

$$D_N = \epsilon_0 E_N = 2.19 \times 10^{-12}(a_x + a_y)$$

$$\rho_s = |D_N| = 3.09 \times 10^{-12}$$

5.5 The Nature of Dielectric Materials

A dielectric in an electric field can be viewed as a *free-space arrangement of microscopic electric dipoles*, each of which is composed of a positive and a negative charge whose centers do not quite coincide. These are *not free charges*, and they cannot *contribute to the conduction process*. Rather, they are bound in place by atomic and molecular forces and can only shift positions slightly in response to external fields. They are called *bound charges*.

The characteristic that all dielectric materials have in common, whether they are solid, liquid, or gas, and whether or not they are crystalline in nature, is their **ability to store electric energy**. This storage takes place by means of a shift in the relative positions of the internal, bound positive and negative charges against the normal molecular and atomic forces.

The dipole may be described by its dipole moment \mathbf{p}

$$\mathbf{P} = Q \mathbf{d}$$

where Q is the positive one of the two bound charges composing the dipole, and \mathbf{d} is the vector from the negative to the positive charge. We note again that the units of \mathbf{p} are coulomb-meters.

There is thus an added term to \mathbf{D} that appears when polarizable material is present

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$$

The linear relationship between \mathbf{P} and \mathbf{E} is

$$\mathbf{P} = \chi_e \epsilon_0 \mathbf{E}$$

where χ_e (chi) is a dimensionless quantity called the *electric susceptibility* of the material.

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \mathbf{E} + \chi_e \epsilon_0 \mathbf{E}$$

$$\mathbf{D} = (\chi_e + 1) \epsilon_0 \mathbf{E}$$

The expression within the parentheses is now defined as

$$\epsilon_r = \chi_e + 1$$

This is another dimensionless quantity, and it is known as the *relative permittivity*, or *dielectric constant* of the material. Thus

$$\mathbf{D} = \epsilon_r \epsilon_0 \mathbf{E} = \epsilon \mathbf{E}$$

$$\epsilon = \epsilon_r \epsilon_0$$

where ϵ is the permittivity

Example Two point charges in a dielectric medium where $\epsilon_r = 5.2$ interact with a force of $8.6 \times 10^{-3} \text{ N}$. What force could be expected if the charges were in free space??

Solution:

$$F_1 = \frac{Q_1 Q_2}{4\pi \epsilon_0 R^2} \quad \text{in free space}$$

$$F_2 = \frac{Q_1 Q_2}{4\pi \epsilon R^2} \quad \text{in dielectric}$$

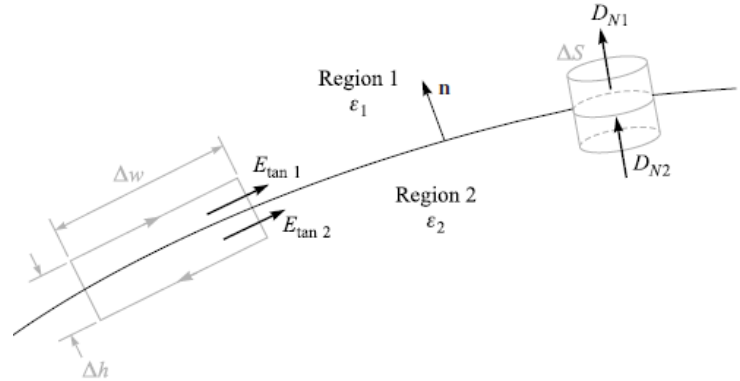
$$\frac{F_1}{F_2} = \frac{\epsilon}{\epsilon_0}$$

$$F_1 = \frac{\epsilon_0 \epsilon_r}{\epsilon_0} F_2 = \epsilon_r F_2 = 5.2 \times 8.6 \times 10^{-3} = 4.47 \times 10^{-2} \text{ N}$$

5.6 Boundary Conditions for Perfect Dielectric Materials

Let us first consider the interface between two dielectrics having permittivities ϵ_1 and ϵ_2 and occupying regions 1 and 2, as shown in Figure below. We first examine the tangential components by using

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$



Around the small closed path on the left, obtaining

$$E_{tan1}\Delta w - E_{tan2}\Delta w = 0$$

The small contribution to the line integral by the normal component of \mathbf{E} along the sections of length Δh becomes negligible as Δh decreases and the closed path crowds the surface. Immediately, then,

$$E_{tan1} = E_{tan2}$$

$$\frac{D_{tan1}}{\epsilon_1} = \frac{D_{tan2}}{\epsilon_2}$$

$$\frac{D_{tan1}}{D_{tan2}} = \frac{\epsilon_1}{\epsilon_2}$$

The boundary conditions on the normal components are found by applying Gauss's law to the small "pillbox" shown at the right in the Figure. The sides are again very short, and the flux leaving the top and bottom surfaces is the difference

$$D_{N1}\Delta S - D_{N2}\Delta S = \Delta Q = \rho_s \Delta S$$

for no free charge is available in the perfect dielectrics, we may assume ρ_s is zero on the interface and

$$D_{N1} = D_{N2}$$

Let \mathbf{D}_1 (and \mathbf{E}_1) make an angle θ_1 with a normal to the surface (Figure below) . Because the normal components of \mathbf{D} are continuous,

$$D_{N1} = D_{N2}$$

$$\therefore D_1 \cos \theta_1 = D_2 \cos \theta_2$$

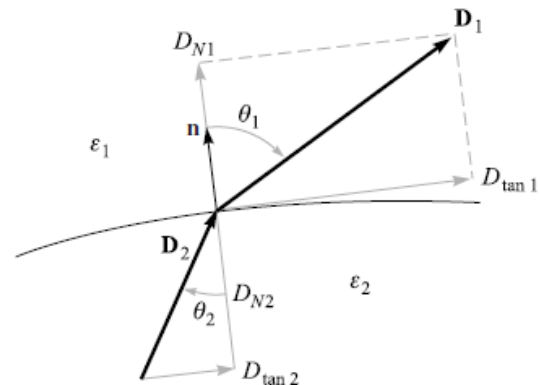
The ratio of the tangential components is given by

$$\frac{D_{\tan 1}}{D_{\tan 2}} = \frac{\epsilon_1}{\epsilon_2} \quad \Leftrightarrow \quad \frac{D_1 \sin \theta_1}{D_2 \sin \theta_2} = \frac{\epsilon_1}{\epsilon_2}$$

$$D_1 \epsilon_2 \sin \theta_1 = D_2 \epsilon_1 \sin \theta_2$$

and the division of these equations gives

$$\frac{\tan \theta_1}{\tan \theta_2} = \frac{\epsilon_1}{\epsilon_2}$$



Example The surface $x = 0$ separates two perfect dielectrics. For $x > 0$, let $\epsilon_{r1} = 3$, while $\epsilon_{r2} = 5$ where $x < 0$. If $\mathbf{E}_1 = 80\mathbf{a}_x - 60\mathbf{a}_y - 30\mathbf{a}_z$ V/m, find (a) E_{N1} ; (b) \mathbf{E}_{T1} ; (c) \mathbf{E}_1 ; (d) the angle θ_1 between \mathbf{E}_1 and a normal to the surface; (e) D_{N2} ; (f) D_{T2} ; (g) \mathbf{D}_2 ; (h) \mathbf{P}_2 ; (i) the angle θ_2 between \mathbf{E}_2 and a normal to the surface.?

Solution:

$$(a) E_{N1} = 80 \text{ ax}$$

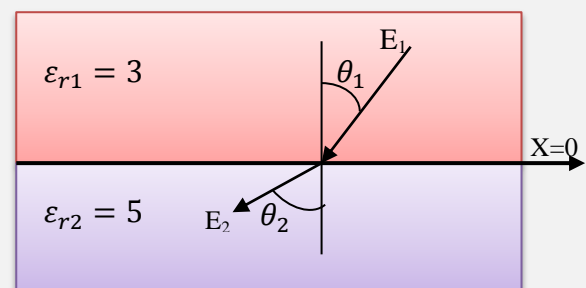
$$(b) \mathbf{E}_{T1} = -60\mathbf{a}_y - 30\mathbf{a}_z$$

$$(c) |\mathbf{E}_1| = \sqrt{80^2 + 60^2 + 30^2} = 104.4 \text{ V/m}$$

$$(d) E_{N1} = E_1 \cos \theta_1$$

$$\cos \theta_1 = \frac{E_{N1}}{E_1} = \frac{80}{104.4}$$

$$\theta_1 = 40^\circ$$



$$(e) D_{N2} = D_{N1}$$

$$D_{N1} = \epsilon_1 E_{N1} = \epsilon_0 \epsilon_{r1} E_{N1} = 3 \times 8.85 \times 10^{-12} \times 80 \text{ ax} = 2.12 \text{ ax nC/m}^2$$

$$(f) E_{T1} = E_{T2}$$

$$\frac{D_{T1}}{\epsilon_1} = \frac{D_{T2}}{\epsilon_2}$$

$$D_{T2} = \frac{\epsilon_2}{\epsilon_1} D_{T1} = \frac{\epsilon_{r2}}{\epsilon_{r1}} \epsilon_1 E_{T1} = \epsilon_{r2} \epsilon_0 E_{T1} = \epsilon_{r2} \epsilon_0 (-60 \text{ ay} - 30 \text{ az})$$

$$D_2 = D_{N2} + D_{T2}$$

$$D_2 = 2.12 \times 10^{-9} \text{ a}_x - 60 \epsilon_{r2} \epsilon_0 \text{ a}_y - 30 \epsilon_{r2} \epsilon_0 \text{ a}_z$$

$$(h) P_2 = (\epsilon_{r2} - 1) \epsilon_0 E_2$$

$$= 4 \epsilon_0 \frac{D_2}{\epsilon_2}$$

$$= 4 \epsilon_0 \frac{D_2}{\epsilon_0 \epsilon_{r2}}$$

$$= 4 \frac{D_2}{\epsilon_{r2}}$$

$$= \frac{4}{5} (2.12 \times 10^{-9} \text{ a}_x - 60 \epsilon_{r2} \epsilon_0 \text{ a}_y - 30 \epsilon_{r2} \epsilon_0 \text{ a}_z)$$

$$(i)$$

$$\frac{\tan \theta_1}{\tan \theta_2} = \frac{\epsilon_1}{\epsilon_2}$$

$$\frac{\tan 40}{\tan \theta_2} = \frac{3}{5}$$

$$\theta_2 = 54.5^\circ$$

Example Region 1, $z < 0$ m, is free space where $D = 5\mathbf{a}_y + 7\mathbf{a}_z$ C/m². Region 2, $0 < z < 1$ m, has $\epsilon_r = 2.5$. And region 3, $z > 1$ m, has $\epsilon_r = 3$. Find E_2 , \mathbf{P}_2 , and θ_3 ?

Solution:

$$D_1 = (5\mathbf{a}_y + 7\mathbf{a}_z)$$

$$D_{N1} = D_1 \cos \theta_1$$

$$\cos \theta_1 = \frac{7}{\sqrt{5^2 + 7^2}}$$

$$\theta_1 = 35.53^\circ$$

$$D_{N2} = D_{N1} = 7\mathbf{a}_z$$

$$E_{T1} = E_{T2}$$

$$\frac{D_{T1}}{\epsilon_1} = \frac{D_{T2}}{\epsilon_2}$$

$$D_{T2} = \frac{\epsilon_2}{\epsilon_1} D_{T1} = \frac{\epsilon_{r2}}{\epsilon_{r1}} D_{T1} = \frac{2.5}{1} 5\mathbf{a}_y = 12.5\mathbf{a}_y$$

$$D_2 = D_{N2} + D_{T2}$$

$$D_2 = 12.5\mathbf{a}_y + 7\mathbf{a}_z$$

$$E_2 = \frac{D_2}{\epsilon_2} = \frac{12.5\mathbf{a}_y + 7\mathbf{a}_z}{2.5\epsilon_0} = \frac{1}{\epsilon_0} \left(5\mathbf{a}_y + \frac{7}{2.5}\mathbf{a}_z \right)$$

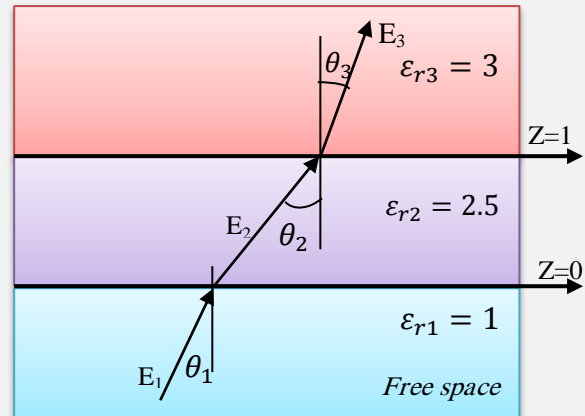
$$D_{N2} = D_2 \cos \theta_2$$

$$\theta_2 = \cos^{-1} \frac{7}{\sqrt{12.5^2 + 7^2}} = 60.7^\circ$$

$$\frac{\tan \theta_2}{\tan \theta_3} = \frac{\epsilon_2}{\epsilon_3}$$

$$\tan \theta_3 = \frac{\epsilon_{r3}}{\epsilon_{r2}} \tan \theta_2 = \frac{3}{2.5} \tan 60.7$$

$$\theta_3 = 64.93^\circ$$



Home work

Q1: Current density is given in cylindrical coordinates as $\mathbf{J} = -10^6 z^{1.5} \mathbf{a}_z$ A/m² in the region $0 \leq \rho \leq 20 \mu\text{m}$; for $\rho \geq 20 \mu\text{m}$, $\mathbf{J} = 0$. Find the total current crossing the surface $z = 0.1$ m?

Ans. $-39.7 \mu\text{A}$

Q1: Given $\mathbf{J} = -10^{-4}(y\mathbf{a}_x + x\mathbf{a}_y)$ A/m², find the current crossing the $y = 0$ plane in the $-\mathbf{a}_y$ direction between $z = 0$ and 1, and $x = 0$ and 2?

Ans: 2×10^{-4} A

Q2: Let Region 1 ($z < 0$) be composed of a uniform dielectric material for which $\epsilon_r = 3.2$, while Region 2 ($z > 0$) is characterized by $\epsilon_r = 2$. Let $\mathbf{D}_1 = -30\mathbf{a}_x + 50\mathbf{a}_y + 70\mathbf{a}_z$ nC/m² and find: (a) D_{N1} ; (b) D_{t1} (c) $|\mathbf{D}_{t1}|$; (d) $D1$; (e) θ_1 ; (f) \mathbf{P}_1 (g) \mathbf{D}_{N2} ; (h) \mathbf{D}_{t2} ; (i) \mathbf{D}_2 ; (k) \mathbf{P}_2 ; (l) θ_2 ?

Ans: 70 nC/m^2 ; $-30\mathbf{a}_x + 50\mathbf{a}_y \text{ nC/m}^2$; 58.3 nC/m^2 ; 91.1 nC/m^2 ; 39.8° ; $-20.6\mathbf{a}_x + 34.4\mathbf{a}_y + 48.1\mathbf{a}_z \text{ nC/m}^2$, $70\mathbf{a}_z \text{ nC/m}^2$; $-18.75\mathbf{a}_x + 31.25\mathbf{a}_y \text{ nC/m}^2$; $-18.75\mathbf{a}_x + 31.25\mathbf{a}_y + 70\mathbf{a}_z \text{ nC/m}^2$; $-9.38\mathbf{a}_x + 15.63\mathbf{a}_y + 35\mathbf{a}_z \text{ nC/m}^2$; 27.5°

Q2: Two perfect dielectrics have relative permittivity $\epsilon_{r1} = 2$ and $\epsilon_{r2} = 8$. The planar interface between them is the surface $x - y + 2z = 5$. The origin lies in region 1. If $\mathbf{E}_1 = 100\mathbf{a}_x + 200\mathbf{a}_y - 50\mathbf{a}_z$ V/m, find \mathbf{E}_2 ?

Ans: $125\mathbf{a}_x + 175\mathbf{a}_y$ V/m

Q2: Figure below shows a planar dielectric slab with free space on either side. Assuming a constant field E_2 within the slab, show that $E_3 = E_1$?

